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# ACCUMULATION OF PERTURBATIONS IN LINEAR AND NON-LINEAR SHOCK-DRIVEN SYSTEMS†

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A dynamical system receives a certain input, whose magnitude has a bounded integral with respect to time; the maximum response of the system to this input is to be determined. The functional representing the response is the maximum of a function of the phase variables, evaluated along a solution of the system. An account is given of types of linear and non-linear systems in which the maximum response is produced, in the limit, by an instantaneous shock. An example is also presented of a non-linear second-order system in which the response to an instantaneous shock is not the maximum possible response to an input of the type described.

ENGINEERS dealing with the design and development of various systems of vibration technology frequently have to determine the maximum response of a system to an input of some previously fixed class. One of the first such problems to be solved was probably Bulgakov's well-known treatment of the accumulation of perturbations in linear systems in which the input is bounded in magnitude [1]. This paper is concerned with the accumulation of perturbations in linear and non-linear systems when the integral of the absolute value of the input with respect to time is bounded.

## 1. GENERAL FORMULATION OF THE PROBLEM

Consider a dynamical system

$$\mathbf{x}^{\cdot} = \mathbf{f}(\mathbf{x}) + \mathbf{b}u, \quad \mathbf{x}(0) = \mathbf{0} \tag{1.1}$$

where $\mathbf{x}$ and $\mathbf{b}$ are $n$-vectors, $\mathbf{f}$ is an $n$-vector-valued function, and $u = u(t)$ is a scalar function representing the input (external action). We define a class of inputs $\Sigma$ as the set of piecewise-continuous functions, defined on the positive real line $[0, \infty)$ that satisfy the integral constraint

$$\int_0^\infty |u(t)| dt \leqslant J_0 \tag{1.2}$$

The performance of transients in system (1.1) may be represented by a functional

$$D[u(\cdot)] = \sup_{t \in [0, \infty)} \Phi[\mathbf{x}(t, u(\cdot))] \tag{1.3}$$

where $\mathbf{x}(t, u(\cdot))$ is the solution of the Cauchy problem for system (1.1), given an input $u(t)$.

We wish to determine the maximum response of the system

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$$D^0 = \sup_{u(\cdot)\in\Sigma} D\,[u(\cdot)] \tag{1.4}$$

Henceforth, by specifying the functions **f** and $\Phi$ in various ways, we shall consider several problems which are special cases of (1.4).

## 2. ACCUMULATION OF PERTURBATIONS IN LINEAR SYSTEMS

Consider the linear system

$$\Phi(\mathbf{x}) = |(\mathbf{a}, \mathbf{x})|. \tag{2.1}$$

Here $A$ is an $n\times n$ matrix and **x** and **b** $n$-vectors. It is assumed that the real parts of the eigenvalues of $A$ are negative, i.e. when there is no input, the linear system is stable. For the function occurring in (1.3) we take the modulus of the scalar product $(\mathbf{a}, \mathbf{x})$, where **a** is an arbitrary constant vector

$$\mathbf{x}^{\cdot} = A\mathbf{x} + \mathbf{b}u, \quad \mathbf{x}(0) = \mathbf{0}.$$

Assuming that $u(t)$ satisfies condition (1.2), consider problem (1.4).

We can write the solution of system (1.2) in the form

$$\mathbf{x}(t) = \int_0^t G(t-\tau)\,\mathbf{b}u(\tau)\,d\tau,$$

where $G(t)$ is the fundamental matrix of solutions of the linear system (2.1) such that $G(0) = E$ ($E$ denotes the $n\times n$ identity matrix).

The scalar product $(\mathbf{a}, \mathbf{x})$ is given by

$$(\mathbf{a}, \mathbf{x}) = \int_0^t \psi(t-\tau)\,u(\tau)\,d\tau, \quad \psi(t) = \sum_{i,j} a_i b_j g_{ij}(t)$$

($a_j$ and $b_j$ are the components of **a** and **b** and $g_{ij}(t)$ are the elements of the matrix $G(t)$). We have

$$|(\mathbf{a}, \mathbf{x})| \leqslant \int_0^t |u(\tau)|\,|\psi(t-\tau)|\,d\tau$$

whence it follows that

$$|(\mathbf{a}, \mathbf{x})| \leqslant p\int_0^t |u(\tau)|\,d\tau \leqslant pJ_0, \quad p = \sup_{t\in[0,\infty)} |\psi(t)|.$$

Note that, since the system is stable, $p$ is bounded. Consider the input

$$u^0(t) = \begin{cases} u_0, & 0\leqslant t\leqslant \tau^* \\ 0, & t>\tau^* \end{cases} \tag{2.2}$$

where $u_0 > 0$, $u_0\tau^* = J_0$. Let us evaluate the integral

$$I(t, \tau^*) = \int_0^t u^0(\tau)\,\psi(t-\tau)\,d\tau.$$

For sufficiently small $\tau^*$, using well-known properties of integrals, we obtain

$$l(t,\tau^*)=\begin{cases} J_0\psi[\alpha\tau^*], & t\leqslant\tau^* \\ J_0\psi[t-\beta\tau^*], & t>\tau^* \end{cases}$$

$$(\alpha=\alpha(t,\tau^*),\ \beta=\beta(t,\tau^*),\ |\alpha|\leqslant 1,\ |\beta|\leqslant 1,\ \alpha(\tau^*,\tau^*)=1-\beta(\tau^*,\tau^*)).$$

The last equality in parentheses follows from the continuity of $\psi(t)$ at $t=\tau^*$.

We have

$$\lim_{\tau^*\to 0} l(t,\tau^*)=J_0\psi(t).$$

Hence, the quantity defined in (1.4) is

$$D^0=J_0\sup_{t\in[0,\infty)}|\psi(t)|.$$

A similar result was obtained for a linear second-order system in [2].

Thus, the input maximizing the performance of the linear system (2.1) is, in the limit, an instantaneous shock of strength $J_0$.

## 3. ACCUMULATION OF PERTURBATIONS IN SECOND-ORDER NON-LINEAR SYSTEMS

Consider a second-order non-linear system

$$x^{\cdot}=y,\ y^{\cdot}=-R(x,y)+u(t);\ x(0)=y(0)=0 \tag{3.1}$$

Problem (1.4) for this system is solvable only when the functions $R$ and $\Phi$ satisfy certain additional conditions. An example is the case considered in [3]; our treatment here is essentially a generalization of those results.

Let

$$R(x,y)=f(x)y+bz(y)+\varphi(x) \tag{3.2}$$

where $f$ and $\varphi$ are functions defined for $x\in(-\infty,\infty)$ and continuous everywhere except possibly at $x=0$, where $\varphi$ may have discontinuities of the first kind. In addition, we assume that

$$f(x)\geqslant 0,\ \varphi(x)\operatorname{sign}(x)\geqslant 0,\ \varphi(x)\not\equiv 0 \tag{3.3}$$
$$\varphi(x_1)\geqslant\varphi(x_2)\ \text{for}\ x_1>x_2,\ x_1\neq 0,\ x_2\neq 0$$

We assume that $b\geqslant 0$; the function $z(y)$, which represents dry friction, is defined for $b\neq 0$ as in [3]

$$z(y)=\begin{cases}\operatorname{sign}(y), & y\neq 0 \\ \nu/b, & y=0,\ |\nu|\leqslant b \\ \operatorname{sign}(\nu), & y=0,\ |\nu|>b\end{cases}$$
$$(\nu=-\varphi(x)+u(t))$$

As to the continuous function $\Phi(x,y)$ describing the performance of the system, we assume that

$$\Phi(x,y_1)\geqslant\Phi(x,y_2)\ \text{for}\ |y_1|>|y_2| \tag{3.4}$$

One instance of a function satisfying this condition is

$$\Phi(x, y) = |x|$$

which is very commonly used in practice, representing the deviation of the system from its initial state.

It follows from relationships (3.2) and (3.3) that, when there is no external input $u(t)$, system (3.1), excited from its initial state and then left to itself, is dissipative (or conservative if $b=0$, $f(x) \equiv 0$). Thus, the value of the function

$$W(x, y) = \frac{y^2}{2} + \int_0^x \varphi(x)\,dx$$

interpreted as the sum of the kinetic and potential energy of the system, does not increase along solutions of the system.

To investigate problem (1.4) with inputs of class $\Sigma$ we will consider pulsed inputs; these comprise the class $L_0$ of all sequences of delta-functions $\delta(t)$ of the type

$$\Delta_n(t) = I_1\,\delta(t-\tau_1) + \ldots + I_n\,\delta(t-\tau_n) \tag{3.5}$$

where $|I_1| + \ldots + |I_n| = J_n \leqslant J_0$, $I_i \neq 0$, $\tau_{i+1} > \tau_i$.

As in [3], one shows that for any input $u(\cdot) \in \Sigma$ there is a pulsed input $\Delta(\cdot) \in L_0$ such that

$$D[\Delta(\cdot)] \geqslant D[u(\cdot)] \tag{3.6}$$

It can be proved that the optimum response (1.4) is

$$D^0 = \max\{D[\Delta^+(\cdot)],\ D[\Delta^-(\cdot)]\};\quad \Delta^{\pm}(t) = \pm J_0\delta(t) \tag{3.7}$$

Considering the $(x, y)$ phase plane, let us consider two trajectories corresponding to single shock inputs of strengths $J_n$ and $-J_n$

$$\Delta_1^{\pm}(t) = \pm J_n\delta(t)$$

We first single out the initial sections of these trajectories that lie in the first and third quadrants, respectively, of the phase plane, denoting them by $F(+J_n)$ and $F(-J_n)$. Suppose that each of these curves is given by an equation

$$y = y_0^{\pm}(x, \pm J_n)$$

Note that $y_0^{\pm}(0, \pm J_n) = \pm J_n$. By [4], it follows from (3.2) and (3.3) that the roots of the equations

$$y_0^{\pm}(x, \pm J_n) = 0$$

which we denote by $\lambda(\pm J_n)$ are finite. Reflect the curves $F(\pm J_n)$ symmetrically about the abscissa axis, into the fourth and second quadrants, respectively (see Fig. 1). This yields a closed curve which we denote by $\Gamma(J_n)$. Denote the domain in the phase plane bounded by $\Gamma(J_n)$ and containing the point (0, 0) by $\mu(J_n)$. Since phase trajectories corresponding to inputs which are single shocks of strengths $J_n$ and $J_{n+1}$ (or $-J_n$ and $-J_{n+1}$) do not intersect, it follows that if $J_{n+1} > J_n$ the phase trajectory corresponding to input

$$\Delta_{m+1}(t) = \Delta_m(t) + I_{m+1}\delta(t-\tau_{m+1})$$

does not leave the domain $\mu^*(J_{n+1})$, where $J_{m+1} = J_m + |I_{m+1}|$. We have [3]

$$y_0^+(x, +J_{m+1}) - y_0^+(x, +J_m) > |I_{m+1}|,\ \forall x \in (0, \lambda(+J_m)]$$

$$|y_0^-(x, -J_{m+1})| - |y_0^-(x, -J_m)| > |I_{m+1}|,\ \forall x \in [\lambda(-J_m), 0)$$

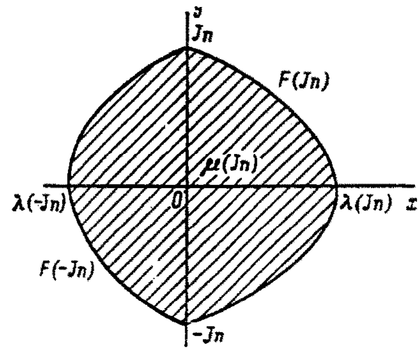


FIG. 1.

Consequently, any point of the phase plane that was in $\mu^*(J_m)$ just before application of the $(m+1)$th shock will remain in $\mu^*(J_{m+1})$ after the shock, and for $t>\tau_{m+1}$ the phase trajectory of the dissipative system corresponding to $\Delta_{m+1}(t)$ will not leave the set $\mu^*(J_{m+1})$. Since $J_m \leqslant J_0$, the phase trajectory for an arbitrary input $\Delta(\cdot)\in L_0$ will not leave the set $\mu^*(J_0)$. By inequality (3.4), for any point $(x_0, y_0)$ of $\mu^*(J_0)$, there exists a point $(x_0^*, y_0^*)$. in either $F(+J_0)$ or $F(-J_0)$, such that

$$\max\{D[\Delta^+(\cdot)], D[\Delta^-(\cdot)]\} \geqslant D[\Delta(\cdot)], \forall \Delta(\cdot)\in L_0$$

Thus, noting (3.6), we obtain

$$\max\{D[\Delta^+(\cdot)], D[\Delta^-(\cdot)]\} \geqslant D[u(\cdot)], \forall u(\cdot)\in \Sigma$$

To prove (3.7), we need only consider an input of type (2.2), which we denote by $u_+^0(t)$ if $u_0>0$, and $u_-^0(t)$ if $u_0<0$ (in either case $|u_0|\tau^*=J_0$). Letting $\tau^*$ tend to zero, we obtain

$$\max\{\lim_{\tau^*\to 0} D[u_+^0(\cdot)], \lim_{\tau^*\to 0} D[u_-^0(\cdot)]\} = \max\{D[\Delta^+(\cdot)], D[\Delta^-(\cdot)]\}$$

We have thus proved that the optimum performance of system (3.1) is produced in the limit by an instantaneous shock of strength $+J_0$ or $-J_0$.

## 4. COUNTEREXAMPLE

The dynamical systems considered in Secs 2 and 3 are examples of systems in which the optimum response to inputs from class $\Sigma$ is evoked by an instantaneous shock. Do systems exist that do not possess this property? We will give a simple counterexample, in which a certain input of class (1.2) produces a response in excess of the response to an instantaneous shock.

Consider the second-order system

$$x^{\cdot} = y,\ y^{\cdot} = -ky|y| - by + u(t);\ x(0) = y(0) = 0 \tag{4.1}$$

This system differs substantially from the non-linear system (3.1) (with conditions (3.2) and (3.3)) in that its right-hand side contains the term $-ky|y|$, which represents quadratic friction. The performance index here will be the maximum deviation

$$D[u(\cdot)] = \sup_{t\in[0,\infty)} |x(t, u(\cdot))|$$

We will first determine the maximum deviation $D_+$ for an instantaneous shock input $\Delta(t) = J_0\delta(t)$. Dividing the second equation of system (4.1) by the first, we get

$$dy/dx = -k\,|y| - b, \quad y(0) = J_0$$

(the action of the shock manifests itself in the new initial conditions). Integrating, we obtain

$$x = -\int_{J_0}^{y} \frac{dy}{k|y|+b}$$

The required maximum deviation $D_+$ is attained at $y = 0$. Consequently

$$D_+ = k^{-1} \ln\left[(kJ_0 + b)/b\right]$$

We will now determine the maximum deviation for an input of type (2.2). Integrating the second equation of system (4.1) over the interval $[0, \tau^*]$ we obtain the inverse relation

$$t(y) = \frac{1}{(b^2 + 4ku_0)^{1/2}} \ln\left[\frac{(y - y_+)y_-}{(y - y_-)y_+}\right] \tag{4.2}$$

$$y_\pm = -\left[b \pm (b^2 + 4ku_0)^{1/2}\right]/(2k)$$

We now find $x^* = x(\tau^*)$. Since $x^\cdot = y$, it follows that

$$x^* = \int_0^{\tau^*} y(t)\,dt = \int_0^{y^*} y\frac{dt}{dy}\,dy$$

where $y^* = y(\tau^*)$. Evaluating the integral, we obtain

$$x^* = \frac{1}{(b^2 + 4ku_0)^{1/2}}\left[y_+ \ln\frac{y^* - y_+}{-y_+} - y_- \ln\frac{y^* - y_-}{-y_-}\right]$$

We now determine $y^*$ from (4.2), remembering that $t(y^*) = \tau^* = J_0/u_0$

$$y^* = y_+ y_- (R-1)/(y_+R - y_-), \quad R = \exp\left[J_0(b^2 + 4ku_0)^{1/2}/u_0\right]$$

Letting $u_0$ tend to zero (the product $u_0\tau^*$ is preserved, remaining equal to zero), we obtain

$$\lim_{u_0\to 0} y^* = 0, \quad \lim_{u_0\to 0} x^* = J_0/b$$

Consequently, the maximum deviation $D^*$ as the response to an input (2.2) of infinitesimal magnitude is $J_0/b$.

Comparing $D_+$ and $D^*$, we find that

$$D_+/D^* = \ln(\alpha+1)/\alpha < 1, \quad (\alpha = kJ_0/b)$$

This example shows that the maximum deviation in system (4.1), as the response to an instantaneous shock, is not the maximum possible response to inputs whose magnitudes have bounded integrals with respect to time.

It would be interesting to ascertain under what conditions the maximum response of a non-linear system to inputs of the class considered above is indeed that produced by an instantaneous shock.

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